Supplement to Geometric Contact Potential

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1 COMPARISON TO OTHER CONTACT PAPERS

In this section, we review a broad collection of computational methods for contact mechanics, with a focus on how the contact constraints are formulated and handled numerically. These methods include contributions from graphics, mechanical engineering, and numerical analysis. Our method is unique in that it simultaneously satisfies all of the following requirements: handles a general class (piecewise smooth) of surfaces, is derived from and consistent with a continuum formulation, requires no contact labeling, handles 3D geometries, is localized, does not cause spurious forces, is differentiable, handles self-contact, and guarantees an intersection-free simulation at all times.

Table 1 summarizes this comparison. The columns are organized as follows:

- Reduction: the method's approach of reducing the nonlinear inequality-constrained problem to a problem or sequence of problems for which an efficient numerical method is available.
- Surface types: the class of surfaces handled by the discretized version of the methods.
- Discrete/continuous: whether the formulation proceeds directly from the discrete case or starts with a continuum model.
- Contact labeling: whether or not the surfaces in contact must be marked a priori.
- 3D: whether or not the method was formulated and implemented in 3D.
- Gap function type: how a method measures the distance to contact for a given configuration.
- Numerical method: the numerical method used to solve the reduced version of the problem, if applicable.
- Strong barrier: If the method is a barrier method, this column indicates if the barrier growth is sufficiently fast for the total potential to diverge as the configuration approaches contact.
- Localization: whether the support of the potential is limited to points at a prescribed distance.
- Spurious rest forces: whether or not there are spurious contact forces at the rest state or close.
- Differentiability: whether or not the method is differentiable with respect to the surface configuration. Restricted differentiability refers to being differentiable only for a subset of configurations (e.g., assuming that the surfaces are sufficiently close to contact).
- Self-contact: whether or not the method handles self-contact.
- Intersection-free: whether or not the scene is guaranteed to remain in an intersection-free configuration at all points on the surface at all times.
- Consistency: if the method begins from a continuum formulation, it describes if the discretized problem converges to the continuum problem in the limit (this does not, in general, imply solution convergence).

In addition, there are several abbreviations used in the table:

- CP: closest point projection.
- DND: distance along the normal direction.
- CPCD: closest point between contact pairs.
- LCP: linear complementarity problem.
- SQP: sequential quadratic programming.
- PL: piecewise linear.
- p.w. smooth: piecewise smooth.
- GS: Gauss-Seidel method.
- PD: projective dynamics.
- XPBD: extended position-based dynamics.

1.1 Comparison Details

[Belytschko and Neal 1991]

- *Gap function*: The distance between spheres centered at sample points on the surface ("pinballs").
- *Intersection-free*: Collision detection is only on pinballs, meaning the finite element meshes may intersect.

[Carpenter et al. 1991]

- *Intersection-free*: The target surface nodes may penetrate the surface (Fig. 1) even if the inequalities are satisfied.

[Taylor and Papadopoulos 1993]

- *Spurious forces at rest*: The contact force is nonzero only for zero distance.

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Table 1. Comparison with prior methods handling collisions.

Reference	Reduction	Surface types	Discrete/continuous	Contact labeling	3D	Gap function type	Numerical method	Strong barrier	Localization	Spurious rest forces	Differentiability	Self-contact	Intersection-free	Consistency
[Belytschko and Neal 1991]	active set	bilinear	C	No	Yes	other*	linear solver	N/A	Yes	No	restricted	Yes	No	Yes
[Carpenter et al. 1991]	active set	PL	D	No	No	CP	GS	N/A	Yes	No	restricted	Not descr.	No	N/A
[Taylor and Papadopoulos 1993]	active set	PL	C	No	No	DND	Newton	N/A	Yes	No	restricted	Not descr.	No	Yes
[Vola et al. 1998]	active set	PL	D	No	No	DND	Lemke's algorithm	N/A	Yes	No	restricted	Not descr.	No	N/A
[Hüeber and Wohlmuth 2006]	active set	PL	C	Yes	Yes	DND	linear solver*	N/A	Yes	No	restricted	Yes	No	Yes
[Popp et al. 2012]	active set	p.w. smooth*	C	Yes	Yes	DND	not described*	N/A	Yes	No	Yes	No	No	Yes
[Razon et al. 2023]	active set	PL	D	No	Yes	CPCD	Newton	N/A	Yes	No	Yes	Yes	Yes	N/A
[Alart and Curnier 1991]	augmented Lagrangian	PL	D	Yes	No	CP	modified/non-smooth Newton*	N/A	Yes	No	Yes	Not descr.	No	N/A
[Simo and Laursen 1992]	augmented Lagrangian	PL	C	No*	No	CP	Newton	N/A	Yes	No	Yes	No	No	Yes
[Wriggers 1995]	augmented Lagrangian	PL	C	No	No	CP	Newton	N/A	Yes	No	Yes	Yes	No	Yes
[Pietzrak and Curnier 1999]	augmented Lagrangian	PL	C	Yes*	Yes	CP	Newton	N/A	Yes	No	Yes	No	No	Yes
[Puso and Laursen 2004]	augmented Lagrangian	PL, bilinear	D	Yes	Yes	other*	Newton	N/A	Yes	No	Yes	Not descr.	No	N/A
[Puso et al. 2008]	augmented Lagrangian	p.w. smooth*	D	Yes	Yes	other*	Newton	N/A	Yes	No	Yes	Not descr.	No	N/A
[Hiermeier et al. 2018]	augmented Lagrangian	smooth*	C	Yes	Yes	other*	Newton	N/A	Yes	No	Yes	Not descr.	No	Yes
[Fernandez et al. 2020]	augmented Lagrangian	PL	C	Yes	Yes	other*	Newton	N/A	Yes	No	Yes	Not descr.	No	Yes
[Daviet 2020]	augmented Lagrangian	PL	D	No	Yes	CPCD	GS/Newton*	N/A	Yes	No	Yes	Yes	No	N/A
[Puso et al. 2024]	augmented Lagrangian	PL, bilinear	D	Yes	Yes	other*	Quasi-Newton	N/A	Yes	No	Yes	Not descr.	No	N/A
[Sauer and De Lorenzis 2013]	augmented Lagrangian, penalty, barrier	PL	C	No	Yes	other*	Newton	Yes	Yes	Yes	Yes	Not descr.	No	Yes
[Christensen et al. 1998]	barrier	PL	D	Yes*	No	DND	Non-smooth Newton; interior point	N/A	Yes	No	restricted	No	No	N/A
[Temizer et al. 2014]	barrier	smooth*	C	Yes	Yes	CP	Newton*	Yes	Yes	No	restricted	No	No	Yes
[Kamensky et al. 2018]	barrier	smooth*	C	No	Yes	Other*	Newton	No	Yes	Yes	Yes	Yes	No	Yes
[Li et al. 2020]	barrier	PL	D	No	Yes	CPCD	Newton	Yes	Yes	Yes	Yes	Yes	Yes	N/A
[Alaydin et al. 2021]	barrier	smooth	C	No	Yes	other*	Newton	Yes	Yes	Yes	Yes	Yes	No	Yes
[Li et al. 2023]	barrier	PL	C	No	Yes	CPCD	Newton	Yes	Yes	Yes	Yes	Yes	Yes	Yes
[Wang et al. 2024]	barrier	PL	C	No	Yes	DND	Newton*	N/A	Yes	No	restricted	Not descr.	No	Yes
[Sassen et al. 2024]	barrier	PL	C	No	Yes	CPCD	Newton	No	No	Yes	Yes	Yes	No	Yes
[Huang et al. 2024]	barrier	PL	D	No	Yes	CPCD	Gauss-Newton	Yes	Yes	Yes	Yes	Yes	Yes	N/A
[Shen et al. 2024]	barrier	PL	D	No	Yes	CPCD	Projected GS	Yes	Yes	Yes	Yes	Yes	No	N/A
[Du et al. 2024]	barrier	PL + implicit*	D	No	Yes	other*	Newton	Yes	Yes	Yes	Yes	Yes	No	N/A
Ours	barrier	p.w. smooth	C	No	Yes	other	Newton	Yes	Yes	No	Yes	Yes	Yes	Yes
[Otaduy et al. 2009]	LCP	PL	D	No	Yes	CPCD*	Projected GS*	N/A	Yes	No	Yes	Yes	No	N/A
[Verschoor and Jalba 2019]	LCP	PL	D	No	Yes	CPCD	QP/Projected CR	N/A	Yes	No	Yes	Yes	No	N/A
[Macklin et al. 2019]	minimum map/Fischer-Burmeister	PL	D	No	Yes	CPCD	Non-smooth Newton	N/A.	Yes	Yes	Yes	Not descr.	No	N/A
[Kloosterman et al. 2001]	modified barrier	PL	D	Yes*	No	DND	Newton*	No	No	Yes	No	No	No	N/A
[Belgacem et al. 1998]	N/A	smooth*	C	Yes	No	DND	N/A	N/A	Yes	No	restricted	No	No	Yes
[Taylor and Wriggers 1999]	N/A	smooth*	C	Yes*	No	CP*	unknown*	N/A	Yes	No	Yes	No	No	Yes
[Larionov et al. 2021]	other	PL + implicit*	D	No	Yes	other*	staggered projections/interior point*	N/A	Yes	No	Yes	No	No	N/A
[Macklin et al. 2020]	PD/XPBD	PL	D	No	Yes	CPCD	Newton*	N/A.	Yes	Yes	Yes	Not descr.	No	N/A
[Benson and Hallquist 1990]	penalty	PL	D	No	Yes	DND*	Newton	N/A	Yes	Yes	Yes	Yes	No	N/A
[Wriggers 1995]	penalty	PL	C	No	No	CP	Newton	N/A	Yes	No	Yes	Yes	No	Yes
[Armero and Petőcz 1998]	penalty	PL	C	No	No	CP	Newton	N/A	Yes	No	restricted	No	No	Yes
[Laursen and Love 2002]	penalty	PL	D	No	Yes	DND	Newton	N/A	Yes	No	restricted	Not descr.	No	N/A
[Kim and Eberle 2022]	penalty	PL	D	No	Yes	CPCD	Newton/Projected CG	N/A	Yes	No	Yes	Yes	No	N/A
[Chen et al. 2024]	penalty	PL	D	No	Yes	CPCD	Other*	N/A	Yes	No	Yes	Yes	No	N/A
[Temizer et al. 2012]	penalty	smooth*	C	Yes	Yes	CP	Newton with Uzawa augmentations	N/A	Yes	No	restricted	No	No	Yes
[Kane et al. 1999]	SQP	PL	C	No	Yes	other*	Other*	N/A	Yes	No	No	Yes	No	Yes
[Deuflhard et al. 2008]	SQP	PL	C	Yes*	No	other*	multigrid with Projected GS*	N/A	Yes	No	Yes	Not descr.	No	Yes
[Youett et al. 2019]	SQP	PL	C	No	Yes	other*	multigrid with Projected GS*	N/A	Yes	No	Yes	Not descr.	No	Yes
[Kaufman et al. 2008]	staggered projections/QP	PL*	D	No	Yes	other*	active set solver for QP*	N/A	Yes	No	Yes	Yes	No	N/A

- *Notes*: Does not describe in detail how the active set is determined, refers to previous work describing the release of contact based on pressure.

[Hüeber and Wohlmuth 2006]

- *Numerical method*: Multigrid is used to solve linear problems. Active set updates are interleaved with multigrid iterations.

[Popp et al. 2012]

- *Numerical method*: Primal-dual active set method.
- *Surface types*: Quadratic finite elements are used.

[Razon et al. 2023]

- *Intersection-free*: The geometrically exact CCD from [Brochu et al. 2012] is used.

[Alart and Curnier 1991]

- *Numerical method*: Solves for primal and dual variables at the same time, instead of alternating.

[Simo and Laursen 1992]

- *Contact labeling*: Contact is assumed to occur between a deformable body and a rigid, immovable obstacle.

[Wriggers 1995]

- *Consistency*: This method is also convergent. An error estimator is given for linear elastic contact problems with small deformations.
- *Notes*: One of several methods described in the book.

[Pietzrak and Curnier 1999]

- *Contact labeling*: The contact surface should be specified, and a bijective mapping is assumed between the surfaces in contact.

[Puso and Laursen 2004]

- *Gap function*: The normal direction at a node is given by averaging the incident face normals. The contact forces are weighted by the overlap of the two contact surfaces when projected to the tangent plane on the non-mortar side.

[Puso et al. 2008]

- *Surface types*: Quadratic finite elements are used.
- *Gap function*: The gap function from [Puso and Laursen 2004] is modified to interpolate quadratic elements into linear subelements.

[Hiermeier et al. 2018]

- *Surface types*: NURBS surfaces and Lagrange basis are supported. A continuous normal field is assumed.
- *Gap function*: The distance between contact nodes along the normal direction of the slave side.
- *Notes*: Explanation of the geometry: https://link.springer.com/article/10.1007/s00466-016-1345-4. This has a discussion of nonconvergence, cycling in particular.

[Fernandez et al. 2020]

- *Gap function*: The distance between contact nodes along the surface normal direction.

[Daviet 2020]

- *Numerical method*: An ADMM version of the augmented Lagrangian is used.
- *Notes*: The possibility of penetration is discussed. The author states that some constraint violations may be desirable in order to avoid excessive artificial strains in certain scenarios. A method for controlling the acceptable amount of contact non-compliance is described. To be specific, an artificial slack variable can be introduced, and some of the contact-induced deformation is felt by the artificial vertex, not the physical one. The amount of acceptable non-compliance is controlled by the relative weight of the true and artificial vertices.

[Puso et al. 2024]

- *Gap function*: The gap function from [Puso and Laursen 2004] is modified to project quadrature points to the non-mortar side along the nodal normal instead of to the closest point.

[Sauer and De Lorenzis 2013]

- *Gap function*: The potentials cannot guarantee that the results are penetration-free (even for barrier-type potentials) and have to be defined for negative distances.
- *Intersection-free*: The barrier method may have intersections due to numerical error (as pointed out in the paper) and lack of CCD.
- *Notes*: This paper presents a unified framework, where different choices of potential correspond to different methods from prior work. This row refers to the barrier method in the paper.

[Christensen et al. 1998]

- *Contact labeling*: A subset of the nodes should be designated as contact nodes.
- *Self-contact*: The algorithm assumes two separate bodies in contact.
- *Consistency*: The convergence of the MP method is discussed, but not under refinement.

[Temizer et al. 2014]

- *Numerical method*: A two-stage Newton scheme is used to reduce the barrier parameter after each iteration.
- *Surface types*: A NURBS discretization of the contact surface is used.
- *Intersection-free*: Without CCD, intersections may happen during the solve. One object with large momentum may pass through the other object during the solve, even though the configuration after the solve is intersection-free.
- *Notes*: The issue of violating geometrical constraints is mentioned: "Within a typical Newton–Raphson type iterative solution of a contact problem based on this method, the incremental update may move the solution outside the feasible region and there is no straightforward procedure for modifying the displacement update to prevent this violation. This is a central challenge."

[Kamensky et al. 2018]

- *Gap function*: The gap function is evaluated on all point pairs (excluding close pairs within a user-specified distance in the rest configuration) in a set of fixed quadrature points within the volume.

- *Barrier*: The discretization's choice of quadrature points means some penetrating configurations where the quadrature points aren't overlapping could have finite potentials.
- *Spurious forces at rest*: A poorly chosen radius of the ball to exclude around a point could lead to spurious forces. In addition, the potential is not orientation-aware, which may also lead to spurious forces.
- *Self-contact*: A fixed-radius ball around a point is excluded.
- *Intersection-free*: For example, if an object is stretched such that its quadrature points are far enough apart, another object may be able to pass through. Additionally, no CCD.

[Li et al. 2020]

- *Spurious forces at rest*: There are examples of spurious forces in the evaluation section of our paper.
- *Intersection-free*: Yes, through barrier potential and CCD.
- *Notes*: "...even small violations of exact contact constraints (which are nonconvex) can lead to impossible to untangle geometric configurations, with a direct impact on physical accuracy and stability."

[Alaydin et al. 2021]

- *Gap function*: See [Kamensky et al. 2018].
- *Spurious forces at rest*: See [Kamensky et al. 2018].
- *Self-contact*: The algorithm excludes a ball around a point from consideration for contact pairs, with a user-defined radius.
- *Intersection-free*: The collision barrier is only defined on a set of fixed quadrature points, meaning the finite element meshes may intersect.

[Li et al. 2023]

- *Spurious forces at rest*: There are examples of spurious forces in the evaluation section of our paper.
- *Intersection-free*: Yes, through barrier potential and CCD.

[Wang et al. 2024]

- *Numerical method*: Interior-point solvers Ipopt and HiOp are used.
- *Intersection-free*: Without CCD, intersections may happen during the solve. One object with large momentum may pass through the other object during the solve, even though the configuration after the solve is intersection-free.

[Sassen et al. 2024]

- *Barrier*: The integral form of the barrier potential converges to infinity when it approaches contact, however, the quadrature only includes each face center, so the barrier is finite when the intersection is not at the face center.
- *Spurious forces at rest*: There are examples of spurious forces in the evaluation section of our paper.
- *Intersection-free*: The barrier is discretized with a single-point quadrature rule at each face center, so the barrier is only infinite when a face center is in contact.
- *Consistency*: The method is also convergent. Convergence studies are included in the paper.

[Huang et al. 2024]

- *Spurious forces at rest*: The method has the same spurious forces as [Li et al. 2020] and [Li et al. 2023].
- *Intersection-free*: Yes, through barrier potential and CCD.

[Shen et al. 2024]

- *Spurious forces at rest*: The method has the same spurious forces as [Li et al. 2020] and [Li et al. 2023].
- *Intersection-free*: CCD is not used. Examples of intersections are shown in the supplemental material.

[Du et al. 2024]

- *Surface types*: The input piecewise linear surface is approximated by an implicit surface, and both representations are used in the collision handling. It's non-trivial to extend to high-order input surfaces.
- *Gap function*: The distance between contact nodes along the surface normal direction. The surface normal is computed on the implicit representation.
- *Spurious forces at rest*: The method has the same spurious forces as [Li et al. 2020] and [Li et al. 2023].
- *Self-contact*: The algorithm excludes a fixed-size ball around a point.
- *Intersection-free*: The CCD is not performed on the implicit surfaces, so nodes may penetrate the implicit surfaces.

[Otaduy et al. 2009]

- *Numerical method*: Newton is used for time stepping. There is an additional algorithmic step to eliminate geometric constraint violations at every step.
- *Gap function*: The distance along the contact normal between contact pairs for edge-edge and vertex-face.
- *Notes*: A contact-free state is ensured by the "constraint manifold refinement."

[Kloosterman et al. 2001]

- *Numerical method*: Augmented Lagrangian style method is used, but with barrier instead of penalty.
- *Contact labeling*: The contact nodes should be specified; otherwise, the barrier includes all pairs of primitives, and adjacent nodes may be pushed apart.
- *Barrier*: The log barrier converges to infinity at a negative distance, and the object may penetrate even more, so a quadratic extrapolation of the log barrier is introduced.
- *Spurious forces at rest*: The log barrier term has global support, which means there's interaction between nodes far apart.
- *Differentiability*: When the closest projection of a node on the contact surface is not unique, the functional is non-differentiable and may cause non-convergence.
- *Self-contact*: The algorithm assumes two separate bodies in contact.
- *Intersection-free*: The log barrier is extrapolated to a quadratic penalty in the object's interior, which means intersections may happen even though a log barrier is used.

[Belgacem et al. 1998]

- *Surface types*: The formulation applies to a general class of finite element bases for surfaces. A well-defined outward unit normal vector is assumed. However, numerical implementation is not discussed.

- *Consistency*: The discrete formulation is consistent with the continuous formulation, and an error estimate between the discrete and true solutions is given.

[Taylor and Wriggers 1999]

- *Numerical method*: The algorithm is implemented in FEAP without mentioning the numerical method used.
- *Surface types*: A smooth normal field on the surface is assumed.
- *Contact labeling*: The contact surface must be specified.
- *Gap function*: It is assumed that the surface is at least locally convex.

[Larionov et al. 2021]

- *Numerical method*: Uses iPopt solver to solve nonlinear subproblems with bound constraints.
- *Surface types*: The input piecewise linear surface is approximated by an implicit surface, and both representations are used in the collision handling. It's non-trivial to extend to high-order input surfaces.
- *Gap function*: The distance between contact nodes along the surface normal direction. The surface normal is computed on the implicit representation.

[Macklin et al. 2020]

- *Numerical method*: A Newton-like method is used.

[Benson and Hallquist 1990]

- *Gap function*: The normal direction at a node is obtained by averaging the incident face normals.
- *Spurious forces at rest*: If an object is thinner than a bucket width, used for contact detection, contact forces could be felt on the opposite side of the surface (e.g., for an inflated balloon). Corners also present problems if the wrong side is picked (Figure 7 in the paper).
- *Differentiability*: In most cases, the method is differentiable. Extremely deformed elements may abruptly leave the local region where contacts are searched for. The element considered in contact with a given node may change sharply if its path transitions between different closest elements.
- *Self-contact*: Explicitly deals with non-smooth contact (does not assume normal existence). In the formulation introduced, an ideal barrier (infinite for admissible) is used. Time-stepping is written as a minimization problem, but the inequality form is returned to and solved using SQP.

[Wriggers 1995]

- *Consistency*: This method is also convergent. An error estimator is given for linear elastic contact problems with small deformations.
- *Notes*: This row refers to the penalty method in the book.

[Armero and Petőcz 1998]

- *Self-contact*: The algorithm assumes two separate bodies in contact, and uses the closest point projection between them in the gap function definition.
- *Notes*: The gap function is updated incrementally during time stepping.

[Kim and Eberle 2022]

- *Notes*: A method to resolve intersections using Global Intersection Analysis is described. They claim failure cases are an inevitable part of production work.

[Chen et al. 2024]

- *Numerical method*: A custom block coordinate descent method based on local Gauss-Seidel iterations is used.
- *Notes*: CCD is performed only for some iterations to save computational costs. The authors acknowledge this may miss collisions but claim it is likely that they will be captured at the next check.

[Temizer et al. 2012]

- *Surface types*: A NURBS discretization of the contact surface is used.

[Kane et al. 1999]

- *Numerical method*: The numerical method is not described in detail; SQP is briefly mentioned.
- *Gap function*: A polynomial form derived from the signed area (volume) associated with contact pairs.
- *Spurious forces at rest*: The contact force is active only when segments intersect.
- *Differentiability*: The contact energy is defined to be infinite for intersecting configurations and zero otherwise. The projection back to the admissible set is also not unique.
- *Notes*: Explicitly deals with non-smooth contact (does not assume normal existence). The formulation introduced uses an ideal barrier (zero for admissible configurations, infinite otherwise).

[Deuflhard et al. 2008]

- *Numerical method*: Multigrid is used to solve QPs with projected Gauss-Seidel applied at different levels.
- *Contact labeling*: A bijective map between the two surfaces in contact is assumed to be given.
- *Gap function*: They assume a one-to-one correspondence between two surfaces, and the distance between corresponding nodes on two contact surfaces is used.
- *Notes*: A bijective mapping between mortar and slave surfaces is assumed.

[Youett et al. 2019]

- *Numerical method*: QPs are solved using a truncated non-smooth Newton multigrid method, with projected Gauss-Seidel applied at different levels.
- *Gap function*: The distance between contact nodes along the surface normal direction is used.

[Kaufman et al. 2008]

- *Numerical method*: A QL active set solver is used.
- *Surface types*: A well-defined outward unit normal vector is assumed.
- *Gap function*: Contact is enforced using velocity constraints. The constraints are based on the relative velocity of contacting points along the normal direction.
- *Notes*: Discusses the possibility of frictional forces creating new contact violations. The predictor-corrector method aims to address this.

2 IMPLEMENTATION DETAILS

Our implementation makes use of most parts of the IPC algorithm, except for the potential formulation. We first utilize the existing broad-phase algorithm to collect contact interactions (edge-vertex in 2D, edge-edge and face-vertex in 3D) within ϵ_{trg}.

Then in 2D, we split edge-vertex interactions into edge-vertex and vertex-vertex pairs, where the former only contain pairs whose minimum distance is not reached at the endpoints of the edge (otherwise the potential vanishes due to mollification). Similarly, in 3D, we split face-vertex interactions into face-vertex, edge-vertex, and vertex-vertex pairs. Face-vertex pairs only contain those whose minimum distance is reached in the interior of the triangle; edge-vertex pairs only contain those whose minimum distance is reached in the interior of the edge. Note that we don't need to further split an edge-edge interaction, since if it reduces to an edge-vertex pair, it's already included in the face-vertex interactions. In total, we have 2 types of collision pairs in 2D and 4 types in 3D. The reason why we have more types than IPC is that the restriction in the distance direction and edge/face orientation requires a finer classification of the contact interactions from the broad phase. More types of pairs do not necessarily lead to a larger number of pairs; in fact, we have shown that the number of pairs is generally smaller than IPC.

Since our formulation allows for a $\epsilon(x)$, i.e., the distance barrier support size, larger than the mesh edge length, simply filtering the interactions by distance is not enough for efficiency, since every vertex has interaction with its 1-ring neighbors. To avoid a significant increase in cost, we first compute the distance types of pairs and evaluate the potential with double precision. For piecewise functions with large ranges of constant values h_ϵ, $H^{\alpha,b}$, δ_α, we cache in which range the variable belongs, to filter pairs with $h_\epsilon = 0$, and to avoid computation of gradient and hessian at trivial values (e.g. when $H^{\alpha,b}(z) = 1$). Note that even if $\epsilon(x)$ is larger than the edge length, it's undesirable to have contact everywhere on the surface (otherwise spurious stresses may appear), so the number of pairs with positive potential values is still sparse on the surface in most cases. We then use auto-generated code from Sympy to compute the gradient and Hessian for every collision pair. We observe that in our examples, the FEM assembly and linear solve are far more expensive than our computation of potential derivatives, so we don't further optimize the efficiency of the potential evaluation.

3 CONVERGENT IPC LIMITATIONS

Here we evaluate the "Convergent IPC" [Li et al. 2023] formulation and compare it to our own.

Li et al. [2023] defines the continuum form of the IPC potential as

$$\psi^{IPC}(x; f) = \max_{y,\ \|x-y\|>r} p_\epsilon^{\text{IPC}}(\|f(x) - f(y)\|)$$

where the barrier function

$$p_\epsilon^{\text{IPC}}(z) = \begin{cases} -\kappa \left(\frac{z}{\epsilon} - 1\right)^2 \ln\left(\frac{z}{\epsilon}\right) & 0 < z < \epsilon \\ 0 & \epsilon \le z \end{cases}$$

and r (with $r \to 0$) is a small radius within which self-contact is ignored. From the above expression, one can make several observations.

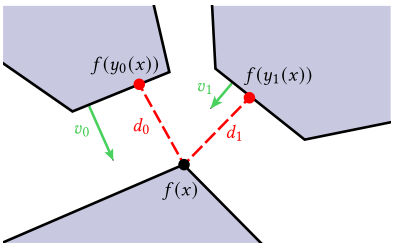

Fig. 1. Here we show the situation for which the dependence of the minimum distance on f is only C^0 continuous for the "Convergent IPC" formulation [Li et al. 2023]. Suppose for $t < t_0$, $d_0 > d_1$ (i.e., d_1 is minimal) and it changes at the constant rate v_1 on some interval $t_s \le t_0 < t_e$. Similarly, d_0 is changing at a higher velocity v_0, but initially is further away, becoming the closest point for $t > t_0$. Although in f_t the dependence of f on t is smooth at every point, the minimal distance does not have a derivative w.r.t. t at this point. If we think in discrete terms, and the endpoints of two segments are q_{ij}, $i, j = 0, 1$, then we can write $d_{\min}(t) = \min(d_0(t), d_1(t))$ as a function $d_{\min}(q_{00} + v_0 t, q_{01} + v_0 t, q_{10} + v_1 t, q_{11} + v_1 t)$, the derivative w.r.t. t is the directional derivative w.r.t. the vector of degrees of freedom q_{ij}, in the direction $[v_{00}, v_{01}, v_{10}, v_{11}]$, and it has a discontinuity.

First, instead of the integral ψ is computed as max; this is not smooth w.r.t. f. This is recognized by Li et al. [2023] with a couple of options proposed (e.g., L_p-norm or LogSumExp) but not implemented. Instead, a smoother approximation is done in the discrete case. Therefore, the method uses a smoother approximation to a non-smooth limit potential. As refinement progresses, the approximation becomes less smooth.

A specific example for which the max is not smooth is when there is a switch between two parametrically distant closest points. Suppose for some x there are two points $f(y_0)$ and $f(y_1)$ equidistant from $f(x)$, and with the distance less than ϵ. If f is time-dependent, i.e., we consider a family f_t, with a scalar parameter t, then at a point y, the velocity $v(y) = \frac{\mathrm{d}}{\mathrm{d}t} f_t(y)$ is defined. If these velocities are different at y_0 and y_1 and the closest point switches from $f(y_0)$ to $f(y_1)$, then the derivative of $\psi(x; f_t)$ has a discontinuity. This is shown in detail in Figure 1.

Second, the choice of r that would ensure that no contacts are missed requires that the curvature of the deformed surface f is bounded from above. Otherwise, no matter how small r is, the surface can fold onto itself, so that there is a contact at a point with $|x - y| < r$. Furthermore, if we would like to use a large ϵ, then r has to be equally large, excluding ever larger parts of Ω, i.e., potentially missing contact.

One smoothed version of $\psi^{IPC}(x; f)$ proposed in the paper uses L_p norm, i.e.,

$$\psi_{L_p}^{IPC}(x; f) = \left(\int_{y,\ \|x-y\|>r} p_\epsilon^{\text{IPC}}(\|f(x) - f(y)\|)^p \right)^{1/p}$$

which is similar to our potential for $p = 1$. However, the interactionset here would be all points excluding a small part of Ω near x determined by r.

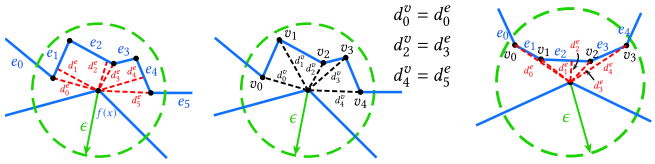

Fig. 2. Discretized convergent IPC geometry. Left: $d_i^e = \|f(x) - f(y_i(x))\|$ shown in red are the distances to the edges that are within ϵ ($\hat{d}$ in IPC notation) of the point $f(x)$, which is assumed to be a vertex. Middle: $d_j^v = \|f(x) - f(x_j)\|$ are distances to vertices within the same radius. Right: the "convex" case that motivates the convergent IPC discretization definition. In this case, the distance from the point $f(x)$ to the other interaction points (i.e., points not on the adjacent edges) that are within the ϵ ball, is increasing as one moves away from the closest point at the distance d_2^e. The distances to all edges other than the closest one are the distances to one of their vertices, which is canceled in the formula, leaving d_2^e only.

The "smooth" discretization proposed for the IPC potential in 2D for a piecewise linear mesh is defined as

$$\psi^{IPC,d}(x;f) = \sum_{\text{edges } i} p_\epsilon^{\text{IPC}}(\|f(x) - f(y_i(x))\|) - \sum_{\text{vertices } j} p_\epsilon^{\text{IPC}}(\|f(x) - f(x_j)\|) \tag{1}$$

where the summation is over all edges not containing x and all vertices excluding x if x is a vertex, and $y_i(x)$ is the closest point to x on the edge i. An illustration of this discretization is given in Figure 2.

Here, one can observe that:

- As $y_i(x)$ may vary nonsmoothly with f, the expression may still be non-smooth w.r.t. f.
- The implicit assumption is that ϵ in p_ϵ^{IPC} is less than any edge length, otherwise, the potential will create an artificial repulsion between adjacent vertices.
- The proof of the discrete potential's positivity (which is not guaranteed by construction) uses the acceptable $\hat{d}$ assumption (ϵ in our notation), which requires either updating ϵ separately at each vertex as the mesh evolves or setting the most conservative ϵ globally, and also updating it as the mesh changes.
- For either the continuum version of the potential or for the discrete version, there is no guarantee that there is no repulsion in the undeformed shape.

4 VERIFICATION OF PROPERTIES

In this section, we outline the verification of the potential properties described in the paper for both the smooth and piecewise smooth formulations. We emphasize that this is not a complete rigorous analysis of the potentials, which we leave as future work.

Several requirements are satisfied for both potentials by construction, so they need not be verified for smooth and piecewise smooth formulations separately.

Requirement 3, *No spurious forces.* This requirement is satisfied by our choice of $\epsilon(x)$ directly, as the potentials are guaranteed to vanish for f_0 (the undeformed shape).

Requirement 4, *Localization.* This is accomplished by the construction of the potential with $\epsilon(x) < \epsilon_{\text{trg}}$.

Requirement 5, *Differentiability.* As the potential integrand is constructed explicitly using compositions of C^1, piecewise C^2 functions, substituting $f(x) = \sum_i p_i B_i(x)$, where $B_i(x)$ are basis functions, and p_i are control points, with respect to which we would like to differentiate Ψ, we observe that the integrand has the desired property, and by the Leibniz integral rule, the integral is differentiable, and piecewise twice differentiable with respect to the parameters p_i.

4.1 Deformable smooth surfaces

We first consider the case where both Ω and $f(\Omega)$ are smooth surfaces and the potential is given by Equation (5).

Proposition 1. *The potential (5) satisfies Requirements 1 to 5 if f is a curvature-continuous surface, with $C(x,f)$ given by Definition 1, distance-to-contact defined as $d_c(x,f) := \min_{y \in C(x,f)} \|f(x) - f(y)\|$, and $\gamma(x,y)$ given by Equation (6), and $p > n-1$, where n is the dimension (2 or 3).*

Requirement 1, *Finiteness.* Each $\psi_\epsilon(x,y;f)$ is an integral of a smooth function defined everywhere on Ω, hence is well defined. It remains to show that it is integrable with respect to x and y, i.e., $\psi_\epsilon(x,y;f)$ is finite. It is sufficient to show that $d_c(x,f)$ is uniformly bounded from below, with respect to x and y, i.e., there is $r > 0$, such that $d_c(x,f) > r$ for any x and y. Then $\psi_\epsilon(x,y;f)$ is uniformly bounded from above, and Ψ is well defined.

In the smooth case, the interaction set $C(x,f)$ is a subset of the set of y satisfying $\Phi^e(x,y) = -n(y)\cdot(f(y)-f(x))_+ \geq -\alpha$. As for a point y approaching x, $f(y)-f(x)$ becomes close to a tangent, in the limit $y \to x$, we have $\Phi^e(x,y) = 1$. $\Phi^e(x,y)$ is a continuous function of y, for each x. Furthermore, due to the curvature continuity assumption, the normal gradient $\nabla_y n(y)$ is continuous on the whole surface, therefore, there is a uniform bound B_n on the norm of the gradient of the normal and a bound B_f on the gradient of f. Consider for all x, a neighborhood of size δ; then the change in Φ^e for a sufficiently small δ, can be bounded, up to a constant, by $\delta(B_f + B_n)$, i.e., for a sufficiently small δ, one can make $\Phi^e(x,y)$ arbitrarily close to 1 for all points in neighborhoods of uniform size δ. In particular, these neighborhoods do not have any points of interaction sets in them. We conclude that $\psi_\epsilon(x,f)$ is bounded and Ψ is well-defined.

Requirement 2, *Barrier.* Consider a family of deformations f_t, $t \in [0,1]$. Suppose f_t is contact-free for $t < 1$ and in contact at $t = 1$, i.e. $f_1(x) = f_1(y)$ for some $x,\ y \in \Omega$ and $x \neq y$. We consider the behavior of the barrier as t approaches 1. The distance function will have local minima around y, which is distinct from x. We can cut out an open disk around x, and let r be the minimal distance to the disk boundary. Consider the remainder of the surface that satisfies the exterior direction constraint. It is non-empty, as it is not empty in the limit $t \to 1$, and Φ^m changes continuously with f_t. On this remainder, the minimum of the distance function is smaller than

r if t is sufficiently close to 1, and this minimum has to be a local minimum, as it is not attained on the surface boundary. We conclude that the interaction set for x cannot be empty, and as the distance to the interaction set approaches zero, the potential becomes infinite, and as the interaction set over which it is integrated, has a limit non-zero measure for $t = 1$, for sufficiently large p (at least 1 for 2D and 2 in 3D), the integral defining the potential $\psi_\epsilon(x, y; f)$ approaches infinity as $t \to 1$, i.e., the barrier property is satisfied.

Consider a smooth contact problem with two surfaces at a distance h, sufficiently small so that the smooth surfaces can be approximated by two parallel planes, and much smaller than ϵ_{trg}. Consider a fixed point x on one plane with the closest point y on the other plane, and a circular patch centered at the point y, of sufficiently small radius r so that it fits into the interaction set. Due to local minimum constraints, $r = kh$, for a $k > 0$ independent of h. This allows us to estimate the inner integral of the potential over the interaction set $C(x, f)$ by the following integral in polar coordinates (ρ, θ) centered at y:

$$\int_\theta \int_\rho \rho (h^2 + \rho^2)^{-p/2} \, \mathrm{d}\rho \mathrm{d}\theta \tag{2}$$

This integral can be computed explicitly and behaves as $\sim h^{2-p}$ for small values of h if $p > 2$. This suggests that the barrier property requires $p > 2$ in 3D, and similarly, $p > 1$ in 2D. As the integrals for all points in a patch of fixed size around x have similar behavior with respect to h, the total potential for the interaction of two fixed-sized patches as $h \to 0$ grows as h^{2-p}.

4.2 Piecewise smooth contact

We now consider the formulation for piecewise smooth surfaces.

Proposition 2. *The potential (14) satisfies Requirements 1 to 5 if f is piecewise smooth surface as defined in the paper, with $C(x, f)$ given by Definition 2, distance-to-contact defined as*

$$d_c(x, f) := \min_{y \in C(x,f)} \|f(x) - f(y)\|,$$

and $\gamma(x, y)$ given by Equation (13).

Requirement 1, *Finiteness.* Each integral $\psi_\epsilon(x, y; f)$ is an integral of a smooth function defined everywhere on Ω, hence is well-defined, as in the smooth case. Unlike the smooth case, however, the distance to the interaction set may not be uniformly bounded from below with respect to x and y. Consider two (possibly curved) faces meeting at an edge curve. Sufficiently close to the edge curve, and near a fixed point on the edge, two faces can be approximated by their tangent planes with cross-sections perpendicular to the edge shown in Figure 3. One can see that the distance to the interaction set can be arbitrarily close to zero. However, the contact area for such points also becomes increasingly small, so if the point-point potential does not grow very quickly, the integrated potential is well-defined.

We assume that the angle γ between the normals of two faces sharing an edge is never zero.

Then for a point at distance r from the edge the distance to the interaction set is $d \sim r \sin\gamma$, and the area of the interaction set, in 3D, $\sim r^2 \sin\theta$. Thus, if potential power $p < 3$, then the integral of

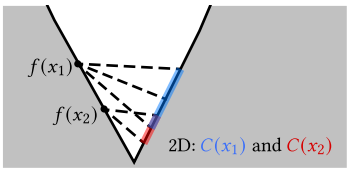

Fig. 3. The interaction set for two points x_1 and x_2 on a concave corner. The $f(x)$ may be arbitrarily close to the apex, but the contact area for such points becomes increasingly small. γ is the angle of the cusp, θ is the angle at $f(x)$ corresponding to the interaction set, $\alpha = \cos\theta$.

the potential over the interaction set for a point at distance r to the edge remains bounded, as $d^{-p} r^2 \sim r^{2-p}$ is integrable. Similarly, in $2D$, the potential needs to grow no faster than $1/r^2$.

Requirement 2, *Barrier.* Consider a family of deformations f_t, $t \in [0, 1]$. Suppose f_t is contact-free for $t < 1$ and in contact at $t = 1$, i.e. $f_1(x) = f_1(y)$ for some $x,\ y \in \Omega$ and $x \neq y$. The fact that for a point x and f sufficiently close to contact the interaction set is not empty is established in the same way as for smooth surfaces. Below we prove that the potential (Equation 14) tends to infinity as $t \to 1$.

- For Vertex-Vertex interactions, no integration is done in (14), and the potential goes to infinity when the distance goes to zero for any positive power p of the potential.
- Suppose one of the primitives is a face, and the distance between primitives is h. Similarly to the smooth case, we can use the estimate of the integral (2), for a potential growth rate for fixed x.
  If the other element is a vertex, no further integration is needed. If the other element is an edge, there is a one-dimensional integral computed along the edge curve, and if we assume that a fixed-size length ℓ along the curve is within distance h of the face, then the total potential is $\sim \ell L \ln(h)$, i.e., it is a barrier, for any $p \geq 2$ ($p \geq 1$ in 2D).
  Note that, in contrast to the smooth case, the range of potentials for 3D, determined by the constraint $p < 3$ for the cusp on the one hand, and $p > 2$, on the other hand, allows for a narrow range of p. We note, however, that the practical effects of these limitations are relatively subtle, as they primarily affect the behavior in the limit, as any potential with a positive p is a barrier in the discrete case, and the limitation at the cusps also only becomes apparent under refinement.
- If one element of the contact pair is an edge, and the other is a vertex or an edge, then the integral for a point on the other element is similar to the integral above, with $\mathrm{d}\rho$ replacing $\rho \mathrm{d}\rho$. The constraint on p is less stringent: any $p > 1$ is sufficient.

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